Chapter 20

Complex numbers

20.1 Cartesian complex numbers

There are several applications of complex numbers in science and engineering, in particular in electrical alternating current theory and in mechanical vector analysis.

There are two main forms of complex number – Cartesian form and polar form – and both are explained in this chapter.

If we can add, subtract, multiply and divide complex numbers in both forms and represent the numbers on an Argand diagram then a.c. theory and vector analysis become considerably easier.

(i) If the quadratic equation \( x^2 + 2x + 5 = 0 \) is solved using the quadratic formula then,

\[
x = \frac{-2 \pm \sqrt{(2)^2 - (4)(1)(5)}}{2(1)}
\]

\[
= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{16(-1)}}{2}
\]

\[
= \frac{-2 \pm 4\sqrt{-1}}{2}
\]

\[
x = -1 \pm 2\sqrt{-1}
\]

It is not possible to evaluate \( \sqrt{-1} \) in real terms. However, if an operator \( j \) is defined as \( j = \sqrt{-1} \) then the solution may be expressed as \( x = -1 \pm j2 \).

(ii) \(-1 + j2 \) and \(-1 - j2 \) are known as complex numbers. Both solutions are of the form \( a + jb \), ‘\( a \)’ being termed the real part and \( jb \) the imaginary part. A complex number of the form \( a + jb \) is called Cartesian complex number.

(iii) In pure mathematics the symbol \( i \) is used to indicate \( \sqrt{-1} \) (\( i \) being the first letter of the word imaginary). However \( i \) is the symbol of electric current in engineering, and to avoid possible confusion the next letter in the alphabet, \( j \), is used to represent \( \sqrt{-1} \).

Problem 1. Solve the quadratic equation \( x^2 + 4 = 0 \).

Since \( x^2 + 4 = 0 \) then \( x^2 = -4 \) and \( x = \sqrt{-4} \).

\[
i.e., \quad x = \sqrt{((-1)(4))} = \sqrt{-1}\sqrt{4} = j(\pm 2)
\]

\[
= \pm j2, \quad \text{since} \quad j = \sqrt{-1}
\]

(Note that \( \pm j2 \) may also be written \( \pm 2j \)).

Problem 2. Solve the quadratic equation \( 2x^2 + 3x + 5 = 0 \).

Using the quadratic formula,

\[
x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(5)}}{2(2)}
\]

\[
= \frac{-3 \pm \sqrt{9 - 40}}{4} = \frac{-3 \pm \sqrt{-31}}{4}
\]

\[
= \frac{-3 \pm j\sqrt{31}}{4}
\]

Hence

\[
x = \frac{-3}{4} \pm \frac{j\sqrt{31}}{4} \quad \text{or} \quad -0.750 \pm j1.392,
\]

correct to 3 decimal places.

(Note, a graph of \( y = 2x^2 + 3x + 5 \) does not cross the \( x \)-axis and hence \( 2x^2 + 3x + 5 = 0 \) has no real roots.)
Problem 3. Evaluate

(a) \( j^3 = j^2 \times j = (-1) \times j = -j \), since \( j^2 = -1 \)
(b) \( j^4 = j^2 \times j^2 = (-1) \times (-1) = 1 \)
(c) \( j^{23} = j \times j^{22} = j \times (j^2)^{11} = j \times (-1)^{11} \)
\( = j \times (-1) = -j \)
(d) \( j^9 = j \times j^8 = j \times (j^2)^4 = j \times (-1)^4 \)
\( = j \times 1 = j \)

Hence \( \frac{-4}{j^9} = \frac{-4}{j} \times \frac{-j}{-j^2} \)
\( = \frac{4j}{-(-1)} = 4j \) or \( 4j \)

Now try the following exercise

Exercise 85 Further problems on the introduction to cartesian complex numbers

In Problems 1 to 9, solve the quadratic equations.

1. \( x^2 + 25 = 0 \) \([\pm 5j]\)
2. \( x^2 - 2x + 2 = 0 \) \([x = 1 \pm j]\)
3. \( x^2 - 4x + 5 = 0 \) \([x = 2 \pm j]\)
4. \( x^2 - 6x + 10 = 0 \) \([x = 3 \pm j]\)
5. \( 2x^2 - 2x + 1 = 0 \) \([x = 0.5 \pm 0.5j]\)
6. \( x^2 - 4x + 8 = 0 \) \([x = 2 \pm 2j]\)
7. \( 25x^2 - 10x + 2 = 0 \) \([x = 0.2 \pm 0.2j]\)
8. \( 2x^2 + 3x + 4 = 0 \)
\( \left[ \frac{-3 \pm j\sqrt{23}}{4} \text{ or } -0.750 \pm j\, 1.199 \right] \)
9. \( 4t^2 - 5t + 7 = 0 \)
\( \left[ \frac{5 \pm j\sqrt{87}}{8} \text{ or } 0.625 \pm j\, 1.166 \right] \)
10. Evaluate (a) \( j^8 \) (b) \( \frac{1}{j^7} \) (c) \( \frac{4}{2j^3} \)
\( [(a) 1 \ (b) - j \ (c) - 2j]\)

20.2 The Argand diagram

A complex number may be represented pictorially on rectangular or cartesian axes. The horizontal (or \( x \)) axis is used to represent the real axis and the vertical (or \( y \)) axis is used to represent the imaginary axis. Such a diagram is called an Argand diagram. In Fig. 20.1, the point \( A \) represents the complex number \( (3 + j2) \) and is obtained by plotting the co-ordinates \( (3, j2) \) as in graphical work. Figure 20.1 also shows the Argand points \( B, C \) and \( D \) representing the complex numbers \( (-2 + j4), (-3 - j5) \) and \( (1 - j3) \) respectively.

\[ \text{Figure 20.1} \]

20.3 Addition and subtraction of complex numbers

Two complex numbers are added/subtracted by adding/subtracting separately the two real parts and the two imaginary parts.

For example, if \( Z_1 = a + jb \) and \( Z_2 = c + jd \),

then \( Z_1 + Z_2 = (a + jb) + (c + jd) \)
\( = (a + c) + j(b + d) \)

and \( Z_1 - Z_2 = (a + jb) - (c + jd) \)
\( = (a - c) + j(b - d) \)
Thus, for example,

\[(2 + j3) + (3 - j4) = 2 + j3 + 3 - j4 = 5 - j1\]

and \[(2 + j3) - (3 - j4) = 2 + j3 - 3 + j4 = -1 + j7\]

The addition and subtraction of complex numbers may be achieved graphically as shown in the Argand diagram of Fig. 20.2. \((2 + j3)\) is represented by vector \(\text{OP}\) and \((3 - j4)\) by vector \(\text{OQ}\). In Fig. 20.2(a) by vector addition (i.e. the diagonal of the parallelogram) \(\text{OP} + \text{OQ} = \text{OR}\). \(R\) is the point \((5, -j1)\).

Hence \((2 + j3) + (3 - j4) = 5 - j1\).

In Fig. 20.2(b), vector \(\text{OQ}\) is reversed (shown as \(\text{OQ}'\)) since it is being subtracted. (Note \(\text{OQ} = 3 - j4\) and \(\text{OQ}' = -(3 - j4) = -3 + j4\). \(\text{OP} - \text{OQ} = \text{OP} + \text{OQ}' = \text{OS}\) is found to be the Argand point \((-1, j7)\).

Hence \((2 + j3) - (3 - j4) = -1 + j7\)

**Problem 4.** Given \(Z_1 = 2 + j4\) and \(Z_2 = 3 - j\) determine (a) \(Z_1 + Z_2\), (b) \(Z_1 - Z_2\), (c) \(Z_2 - Z_1\) and show the results on an Argand diagram.

(a) \(Z_1 + Z_2 = (2 + j4) + (3 - j) = (2 + 3) + j(4 - 1) = 5 + j3\)

(b) \(Z_1 - Z_2 = (2 + j4) - (3 - j) = (2 - 3) + j(4 - (-1)) = -1 + j5\)

(c) \(Z_2 - Z_1 = (3 - j) - (2 + j4) = (3 - 2) + j(-1 - 4) = 1 - j5\)

Each result is shown in the Argand diagram of Fig. 20.3.
20.4 Multiplication and division of complex numbers

(i) Multiplication of complex numbers is achieved by assuming all quantities involved are real and then using \( j^2 = -1 \) to simplify.

Hence \((a + jb)(c + jd)\)

\[
= ac + a(jd) + j(b)c + (j b)(j d)
= ac + j ad + j bc + j^2 bd
= (ac - bd) + j (ad + bc),
\]

since \( j^2 = -1 \)

Thus \((3 + j 2)(4 - j 5)\)

\[
= 12 - j 15 + j 8 - j^2 10
= (12 - (-10)) + j (-15 + 8)
= 22 - j 7
\]

(ii) The complex conjugate of a complex number is obtained by changing the sign of the imaginary part. Hence the complex conjugate of \(a + jb\) is \(a - jb\). The product of a complex number and its complex conjugate is always a real number.

For example,

\[(3 + j 4)(3 - j 4) = 9 - j 12 + j 12 - j^2 16
= 9 + 16 = 25\]

\[(a + jb)(a - jb)\] may be evaluated ‘on sight’ as \(a^2 + b^2\).

(iii) Division of complex numbers is achieved by multiplying both numerator and denominator by the complex conjugate of the denominator.

For example,

\[
\frac{2 - j 5}{3 + j 4} = \frac{2 - j 5}{3 + j 4} \times \frac{3 - j 4}{3 - j 4}
= \frac{6 - j 8 - j 15 + j^2 20}{3^2 + 4^2}
= \frac{-14 - j 23}{25} = \frac{-14}{25} - \frac{j 23}{25}
\]

or \(-0.56 - j 0.92\)

**Problem 5.** If \(Z_1 = 1 - j 3\), \(Z_2 = -2 + j 5\) and \(Z_3 = -3 - j 4\), determine in \(a + jb\) form:

(a) \(Z_1Z_2\)  

(b) \(Z_1\)  

(c) \(\frac{Z_1Z_2}{Z_1 + Z_2}\)  

(d) \(Z_1Z_2Z_3\)

(a) \(Z_1Z_2 = (1 - j 3)(-2 + j 5)\)

\[
= -2 + j 5 + 6 - j^2 15
= (-2 + 15) + j(5 + 6), \text{ since } j^2 = -1,
= 13 + j 11
\]

(b) \(\frac{Z_1}{Z_3} = \frac{1 - j 3}{-3 - j 4} \times \frac{-3 + j 4}{-3 + j 4} = \frac{-3 + j 4 + j 9 - j^2 12}{3^2 + 4^2} = \frac{9 + j 13}{25} = \frac{9}{25} + j \frac{13}{25}
\]

or \(0.36 + j 0.52\)

(c) \(\frac{Z_1Z_2}{Z_1 + Z_2} = \frac{(1 - j 3)(-2 + j 5)}{(1 - j 3) + (-2 + j 5)}\)

\[
= \frac{13 + j 11}{-1 + j 2}, \text{ from part (a),}
= \frac{13 + j 11}{-1 + j 2} \times \frac{-1 - j 2}{-1 - j 2}
= \frac{-13 - j 26 - j 11 - j^2 22}{1^2 + 2^2}
= \frac{9 - j 37}{5} = \frac{9}{5} - j \frac{37}{5} \text{ or } 1.8 - j 7.4
\]

(d) \(Z_1Z_2Z_3 = (13 + j 11)(-3 - j 4), \text{ since}\)

\(Z_1Z_2 = 13 + j 11, \text{ from part (a)}\)

\[
= -39 - j 52 - j 33 - j^2 44
= (-39 + 44) - j(52 + 33)
= 5 - j 85
\]

**Problem 6.** Evaluate:

(a) \(\frac{2}{(1 + j)^4}\)  

(b) \(j \left(\frac{1 + j 3}{1 - j 2}\right)^2\)
(a) \((1 + j)^2 = (1 + j)(1 + j) = 1 + j + j + j^2 = 1 + j + 1 = j2\)

\((1 + j)^4 = [(1 + j)^2]^2 = (j2)^2 = j^4 = -4\)

Hence \(\frac{2}{1 + j} = \frac{2}{-4} = -\frac{1}{2}\)

(b) \(\frac{1 + j3}{1 - j2} = \frac{1 + j3}{1 + j2} \times \frac{1 + j2}{1 + j2} = \frac{1 + j2 + j3 + j^26}{1^2 + 2^2} = \frac{-5 + j5}{5} = -1 + j1 = -1 + j\)

\((\frac{1 + j3}{1 - j2})^2 = (-1 + j)^2 = (-1 + j)(-1 + j) = 1 - j - j + j^2 = -j2\)

Hence \(-j\left(\frac{1 + j3}{1 - j2}\right)^2 = j(-j2) = -j^22 = 2\),

since \(j^2 = -1\)

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**Exercise 86** Further problems on operations involving Cartesian complex numbers

1. Evaluate (a) \((3 + j2) + (5 - j)\) and \((b) (-2 + j6) - (3 - j2)\) and show the results on an Argand diagram.

\[[(a) 8 + j \quad (b) -5 + j8]\]

2. Write down the complex conjugates of (a) \(3 + j4\), (b) \(2 - j\).

\[[(a) 3 - j4 \quad (b) 2 + j]\]

3. If \(z = 2 + j\) and \(w = 3 - j\) evaluate (a) \(z + w\) (b) \(w - z\) (c) \(3z - 2w\) (d) \(5z + 2w\) (e) \(j(2w - 3z)\) (f) \(2jw - jz\)

\[[(a) 5 \quad (b) 1 - j2 \quad (c) j5 \quad (d) 16 + j3 \quad (e) 5 \quad (f) 3 + j4]\]

In Problems 4 to 8 evaluate in \(a + jb\) form given \(Z_1 = 1 + j2, \ Z_2 = 4 - j3, \ Z_3 = -2 + j3\) and \(Z_4 = -5 - j\).

4. (a) \(Z_1 + Z_2 - Z_3\) (b) \(Z_2 - Z_1 + Z_4\)

\[[(a) 7 - j4 \quad (b) -2 - j6]\]

5. (a) \(Z_1Z_2\) (b) \(Z_3Z_4\)

\[[(a) 10 + j5 \quad (b) 13 - j13]\]

6. (a) \(Z_1Z_3 + Z_4\) (b) \(Z_1Z_2Z_3\)

\[[(a) -13 - j2 \quad (b) -35 + j20]\]

7. (a) \(\frac{Z_1}{Z_2}\) (b) \(\frac{Z_1 + Z_3}{Z_2 - Z_4}\)

\[[(a) \frac{-2 + j11}{25} \quad (b) \frac{-19}{85} + j\frac{43}{85}]\]

8. (a) \(\frac{Z_1Z_3}{Z_1 + Z_3}\) (b) \(Z_2 + \frac{Z_1}{Z_4 + Z_3}\)

\[[(a) \frac{3}{26} + j\frac{41}{26} \quad (b) \frac{45}{26} - j\frac{9}{26}]\]

9. Evaluate (a) \(\frac{1 - j}{1 + j}\) (b) \(\frac{1}{1 + j}\)

\[[(a) - j \quad (b) \frac{1}{2} - j\frac{1}{2}]\]

10. Show that \(-\frac{25}{2} \left(\frac{1 + j2}{3 + j4} - \frac{2 - j5}{-j}\right)\)

\[= 57 + j24\]

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### 20.5 Complex equations

If two complex numbers are equal, then their real parts are equal and their imaginary parts are equal. Hence if \(a + jb = c + jd\), then \(a = c\) and \(b = d\).

**Problem 7.** Solve the complex equations:

(a) \(2(x + jy) = 6 - j3\)

(b) \((1 + j2)(-2 - j3) = a + jb\)

(a) \(2(x + jy) = 6 - j3\) hence \(2x + j2y = 6 - j3\)

Equating the real parts gives:

\[2x = 6, \ i.e. x = 3\]

Equating the imaginary parts gives:

\[2y = -3, \ i.e. y = -\frac{3}{2}\]
Problem 8. Solve the equations:

(a) \((2 - j3) = \sqrt{(a + jb)}\)

Hence \((2 - j3)^2 = a + jb\),
i.e. \((2 - j3)(2 - j3) = a + jb\)

Hence \(4 - j6 - j6 = a + jb\)

and \(-5 - j12 = a + jb\)

Thus \(a = -5\) and \(b = -12\)

(b) \((x - j2y) + (y - j3x) = 2 + j3\)

Hence \((x + y) + j(-2y - 3x) = 2 + j3\)

Equating real and imaginary parts gives:
\[x + y = 2\] (1)
\[-3x - 2y = 3\] (2)

i.e. two simultaneous equations to solve.

Multiplying equation (1) by 2 gives:
\[2x + 2y = 4\] (3)

Adding equations (2) and (3) gives:
\[-x = 7\text{, i.e.}, x = -7\]

From equation (1), \(y = 9\), which may be checked in equation (2).

Exercise 87  Further problems on complex equations

In Problems 1 to 4 solve the complex equations.

1. \((2 + j)(3 - j2) = a + jb\) \([a = 8, b = -1]\)

2. \(\frac{2 + j}{1 - j} = j(x + jy)\) \([x = \frac{3}{2}, y = -\frac{1}{2}]\)

3. \((2 - j3) = \sqrt{(a + jb)}\) \([a = -5, b = -12]\)

4. \((x - j2y) - (y - jx) = 2 + j\) \([x = 3, y = 1]\)

5. If \(Z = R + j\omega L + 1/j\omega C\), express \(Z\) in \((a + jb)\) form when \(R = 10\), \(L = 5\), \(C = 0.04\) and \(\omega = 4\).
\([Z = 10 + j13.75]\)

20.6 The polar form of a complex number

(i) Let a complex number \(z\) be \(x + jy\) as shown in the Argand diagram of Fig. 20.4. Let distance \(OZ\) be \(r\) and the angle \(OZ\) makes with the positive real axis be \(\theta\).

From trigonometry, \(x = r\cos \theta\) and \(y = r\sin \theta\)

Hence \(Z = x + jy = r\cos \theta + jr\sin \theta\)
\[= r(\cos \theta + j\sin \theta)\]

\(Z = r(\cos \theta + j\sin \theta)\) is usually abbreviated to \(Z = r\angle \theta\) which is known as the polar form of a complex number.

(ii) \(r\) is called the modulus (or magnitude) of \(Z\) and is written as mod \(Z\) or \(|Z|\).
\(r\) is determined using Pythagoras’ theorem on triangle \(OAZ\) in Fig. 20.4,
\[r = \sqrt{x^2 + y^2}\]
(iii) $\theta$ is called the **argument** (or amplitude) of $Z$ and is written as $\arg Z$.

By trigonometry on triangle $OAZ$,

$$\arg Z = \theta = \tan^{-1} \frac{y}{x}$$

(iv) Whenever changing from cartesian form to polar form, or vice-versa, a sketch is invaluable for determining the quadrant in which the complex number occurs.

**Problem 9.** Determine the modulus and argument of the complex number $Z = 2 + j3$, and express $Z$ in polar form.

$Z = 2 + j3$ lies in the first quadrant as shown in Fig. 20.5.

![Figure 20.5](image)

- **Modulus**, $|Z| = r = \sqrt{(2^2 + 3^2)} = \sqrt{13}$ or $3.606$, correct to 3 decimal places.
- **Argument**, $\arg Z = \theta = \tan^{-1} \frac{3}{2}

  = 56.31^\circ$ or $56^\circ 19'$. 

In polar form, $2 + j3$ is written as $3.606 \angle 56.31^\circ$.

**Problem 10.** Express the following complex numbers in polar form:

(a) $3 + j4$    (b) $-3 + j4$  
(c) $-3 - j4$    (d) $3 - j4$

- (a) $3 + j4$ is shown in Fig. 20.6 and lies in the first quadrant.
  - Modulus, $r = \sqrt{(3^2 + 4^2)} = 5$ and argument $\theta = \tan^{-1} \frac{4}{3} = 53.13^\circ$.
  - Hence $3 + j4 = 5 \angle 53.13^\circ$

- (b) $-3 + j4$ is shown in Fig. 20.6 and lies in the second quadrant.
  - Modulus, $r = 5$ and angle $\alpha = 53.13^\circ$, from part (a).
  - Argument $= 180^\circ - 53.13^\circ = 126.87^\circ$ (i.e. the argument must be measured from the positive real axis).
  - Hence $-3 + j4 = 5 \angle 126.87^\circ$

- (c) $-3 - j4$ is shown in Fig. 20.6 and lies in the third quadrant.
  - Modulus, $r = 5$ and angle $\alpha = 53.13^\circ$, as above.
  - Hence the argument $= 180^\circ + 53.13^\circ = 233.13^\circ$, which is the same as $-126.87^\circ$.
  - **Hence** $(-3 - j4) = 5 \angle 233.13^\circ$ or $5 \angle -126.87^\circ$

  (By convention the **principal value** is normally used, i.e. the numerically least value, such that $-\pi < \theta < \pi$).

- (d) $3 - j4$ is shown in Fig. 20.6 and lies in the fourth quadrant.
  - Modulus, $r = 5$ and angle $\alpha = 53.13^\circ$, as above.
  - **Hence** $(3 - j4) = 5 \angle -53.13^\circ$

**Problem 11.** Convert (a) $4 \angle 30^\circ$ (b) $7 \angle -145^\circ$ into $a + jb$ form, correct to 4 significant figures.

(a) $4 \angle 30^\circ$ is shown in Fig. 20.7(a) and lies in the first quadrant.
20.7 Multiplication and division in polar form

If \( Z_1 = r_1 \angle \theta_1 \) and \( Z_2 = r_2 \angle \theta_2 \) then:

(i) \( Z_1 Z_2 = r_1 r_2 \angle (\theta_1 + \theta_2) \) and

(ii) \( \frac{Z_1}{Z_2} = \frac{r_1}{r_2} \angle (\theta_1 - \theta_2) \)

Problem 12. Determine, in polar form:
(a) \( 8 \angle 25^\circ \times 4 \angle 60^\circ \)
(b) \( 3 \angle 16^\circ \times 5 \angle -44^\circ \times 2 \angle 80^\circ \)

(a) \( 8 \angle 25^\circ \times 4 \angle 60^\circ = (8 \times 4) \angle (25^\circ + 60^\circ) = 32 \angle 85^\circ \)
(b) \( 3 \angle 16^\circ \times 5 \angle -44^\circ \times 2 \angle 80^\circ = (3 \times 5 \times 2) \angle [16^\circ + (-44^\circ) + 80^\circ] = 30 \angle 52^\circ \)

Problem 13. Evaluate in polar form
(a) \( \frac{16 \angle 75^\circ}{2 \angle 15^\circ} \)
(b) \( \frac{10 \angle \frac{\pi}{4} \times 12 \angle \frac{\pi}{2}}{6 \angle -\frac{\pi}{3}} \)

(a) \( \frac{16 \angle 75^\circ}{2 \angle 15^\circ} = \frac{16}{2} \angle (75^\circ - 15^\circ) = 8 \angle 60^\circ \)
(b) \( \frac{10 \angle \frac{\pi}{4} \times 12 \angle \frac{\pi}{2}}{6 \angle -\frac{\pi}{3}} = \frac{10 \times 12}{6} \angle \left(\frac{\pi}{4} + \frac{\pi}{2} - \left(-\frac{\pi}{3}\right)\right) = 20 \angle \frac{13\pi}{12} \) or \( 20 \angle -\frac{11\pi}{12} \) or \( 20 \angle 195^\circ \) or \( 20 \angle -165^\circ \)

Problem 14. Evaluate, in polar form
\( 2 \angle 30^\circ + 5 \angle -45^\circ - 4 \angle 120^\circ \).

Addition and subtraction in polar form is not possible directly. Each complex number has to be converted into cartesian form first.

\( 2 \angle 30^\circ = 2(\cos 30^\circ + j \sin 30^\circ) \)

\( = 2 \cos 30^\circ + j \sin 30^\circ = 1.732 + j 0.866 \)

\( 5 \angle -45^\circ = 5(\cos (-45^\circ) + j \sin (-45^\circ)) \)

\( = 5 \cos (-45^\circ) + j 5 \sin (-45^\circ) \)

\( = 3.536 - j 3.536 \)

\( 4 \angle 120^\circ = 4(\cos 120^\circ + j \sin 120^\circ) \)

\( = 4 \cos 120^\circ + j 4 \sin 120^\circ \)

\( = -2.000 + j 3.464 \)

Hence \( 2 \angle 30^\circ + 5 \angle -45^\circ - 4 \angle 120^\circ \)
Now try the following exercise

Exercise 88  Further problems on polar form

1. Determine the modulus and argument of (a) $2 + j4$ (b) $-5 - j2$ (c) $j(2 - j)$.
   
   (a) $4.472, 63.43^\circ$
   
   (b) $5.385, -158.20^\circ$
   
   (c) $2.236, 63.43^\circ$

   In Problems 2 and 3 express the given Cartesian complex numbers in polar form, leaving answers in surd form.

2. (a) $2 + j3$ (b) $-4$ (c) $-6 + j$
   
   (a) $\sqrt{13} \angle 56.31^\circ$ (b) $4 \angle 180^\circ$
   
   (c) $\sqrt{37} \angle 170.54^\circ$

3. (a) $-j3$ (b) $(-2 + j)^3$ (c) $j^3(1 - j)$
   
   (a) $3 \angle -90^\circ$ (b) $\sqrt{125} \angle 100.30^\circ$
   
   (c) $\sqrt{2} \angle -135^\circ$

In Problems 4 and 5 convert the given polar complex numbers into $(a + jb)$ form giving answers correct to 4 significant figures.

4. (a) $5 \angle 30^\circ$ (b) $3 \angle 60^\circ$ (c) $7 \angle 45^\circ$
   
   (a) $4.330 + j2.500$
   
   (b) $1.500 + j2.598$
   
   (c) $4.950 + j4.950$

5. (a) $6 \angle 125^\circ$ (b) $4 \angle \pi$ (c) $3.5 \angle -120^\circ$
   
   (a) $-3.441 + j4.915$
   
   (b) $-4.000 + j0$
   
   (c) $-1.750 - j3.031$

In Problems 6 to 8, evaluate in polar form.

6. (a) $3 \angle 20^\circ \times 15 \angle 45^\circ$
   
   (b) $2.4 \angle 65^\circ \times 4.4 \angle -21^\circ$

   [a] $45 \angle 65^\circ$ (b) $10.56 \angle 44^\circ$

7. (a) $6.4 \angle 27^\circ \div 2 \angle -15^\circ$
   
   (b) $5 \angle 30^\circ \times 4 \angle 80^\circ \div 10 \angle -40^\circ$

   [a] $3.2 \angle 42^\circ$ (b) $2 \angle 150^\circ$

8. (a) $4 \angle \frac{\pi}{6} + 3 \angle \frac{\pi}{8}$
   
   (b) $2 \angle 120^\circ + 5.2 \angle 58^\circ - 1.6 \angle -40^\circ$

   [a] $6.986 \angle 26.79^\circ$ (b) $7.190 \angle 85.77^\circ$

20.8 Applications of complex numbers

There are several applications of complex numbers in science and engineering, in particular in electrical alternating current theory and in mechanical vector analysis.

The effect of multiplying a phasor by $j$ is to rotate it in a positive direction (i.e. anticlockwise) on an Argand diagram through $90^\circ$ without altering its length. Similarly, multiplying a phasor by $-j$ rotates the phasor through $-90^\circ$. These facts are used in a.c. theory since certain quantities in the phasor diagrams lie at $90^\circ$ to each other. For example, in the $R-L$ series circuit shown in Fig. 20.8(a), $V_L$ leads $I$ by $90^\circ$ (i.e. $I$ lags $V_L$ by $90^\circ$) and may be written as $jV_L$, the vertical axis being regarded as the imaginary axis of an Argand diagram. Thus $V_R + jV_L = V$ and since $V_R = IR, V = jX_L$ (where $X_L$ is the inductive reactance, $2\pi fL$ ohms) and $V = ZI$ (where $Z$ is the impedance) then $R + jX_L = Z$.

Figure 20.8
Similarly, for the \( R-C \) circuit shown in Fig. 20.8(b), \( V_C \) lags \( I \) by 90° (i.e. \( I \) leads \( V_C \) by 90°) and \( V_R \) \(- j V_C = V \), from which \( R - j X_C = Z \) (where \( X_C \) is the capacitive reactance \( \frac{1}{2\pi f C} \) ohms).

**Problem 15.** Determine the resistance and series inductance (or capacitance) for each of the following impedances, assuming a frequency of 50 Hz:

(a) \((4.0 + j 7.0) \Omega \) \( \quad \) (b) \(- j 20 \Omega \)

(c) \(15\angle -60^\circ \) \( \Omega \)

(a) Impedance, \( Z = (4.0 + j 7.0) \Omega \) hence, resistance = 4.0 \( \Omega \) and reactance = 7.00 \( \Omega \). Since the imaginary part is positive, the reactance is inductive, i.e. \( X_L = 7.0 \Omega \).

Since \( X_L = 2\pi f L \) then **inductance**, 
\[
L = \frac{X_L}{2\pi f} = \frac{7.0}{2\pi (50)} = 0.0223 \text{H or 22.3 mH}
\]

(b) Impedance, \( Z = j 20 \), i.e. \( Z = (0 - j 20) \Omega \) hence resistance = 0 and reactance = 20 \( \Omega \). Since the imaginary part is negative, the reactance is capacitive, i.e., \( X_C = 20 \Omega \) and since \( X_C = \frac{1}{2\pi f C} \) then:

**capacitance**, \( C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi (50)(20)} F \)
\[
= \frac{10^6}{2\pi (50)(20)} \mu F = 159.2 \mu F
\]

(c) Impedance, \( Z = 15\angle -60^\circ = 15[\cos (-60^\circ) + j \sin (-60^\circ)] \)
\[
= 7.50 - j 12.99 \Omega
\]

Hence resistance = 7.50 \( \Omega \) and capacitive reactance, \( X_C = 12.99 \Omega \).

Since \( X_C = \frac{1}{2\pi f C} \) then **capacitance**, 
\[
C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi (50)(12.99)} \mu F
\]
\[
= 245 \mu F
\]

**Problem 16.** An alternating voltage of 240 V, 50 Hz is connected across an impedance of \((60 - j 100) \Omega \). Determine (a) the resistance (b) the capacitive reactance \( X_C = 100 \Omega \) and since \( X_C = \frac{1}{2\pi f C} \) then:

**capacitance**, \( C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi (50)(100)} \mu F \)
\[
= \frac{10^6}{2\pi (50)(100)} \mu F = 31.83 \mu F
\]

(c) Magnitude of impedance,
\[
|Z| = \sqrt{[(60)^2 + (-100)^2]} = 116.6 \Omega
\]

Phase angle, \( \arg Z = \tan^{-1} \left( \frac{-100}{60} \right) = -59.04^\circ \)

(d) Current flowing, \( I = \frac{V}{Z} = \frac{240\angle 0^\circ}{116.6\angle -59.04^\circ} \)
\[
= 2.058 \angle 59.04^\circ A
\]

The circuit and phasor diagrams are as shown in Fig. 20.8(b).

**Problem 17.** For the parallel circuit shown in Fig. 20.9, determine the value of current \( I \) and its phase relative to the 240 V supply, using complex numbers.

![Figure 20.9](image-url)
Current \( I = \frac{V}{Z} \). Impedance \( Z \) for the three-branch parallel circuit is given by:

\[
\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3},
\]

where \( Z_1 = 4 + j3 \), \( Z_2 = 10 \) and \( Z_3 = 12 - j5 \)

Admittance, \( Y_1 = \frac{1}{Z_1} = \frac{1}{4 + j3} \)

\[
= \frac{1}{4 + j3} \times \frac{4 - j3}{4 - j3} = \frac{4 - j3}{4^2 + 3^2} = 0.160 - j0.120 \text{ siemens}
\]

Admittance, \( Y_2 = \frac{1}{Z_2} = \frac{1}{10} = 0.10 \text{ siemens} \)

Admittance, \( Y_3 = \frac{1}{Z_3} = \frac{1}{12 - j5} \)

\[
= \frac{1}{12 - j5} \times \frac{12 + j5}{12 + j5} = \frac{12 + j5}{12^2 + 5^2} = 0.0710 + j0.0296 \text{ siemens}
\]

Total admittance, \( \frac{1}{Y} = \frac{1}{Y_1} + \frac{1}{Y_2} + \frac{1}{Y_3} \)

\[
= (0.160 - j0.120) + (0.10) + (0.0710 + j0.0296) = 0.331 - j0.0904 = 0.343\angle -15.28^\circ \text{ siemens}
\]

Current \( I = \frac{V}{Z} = V\frac{1}{Y} \)

\[
= (240\angle0^\circ)(0.343\angle-15.28^\circ) = 82.32\angle-15.28^\circ \text{ A}
\]

**Problem 18.** Determine the magnitude and direction of the resultant of the three coplanar forces given below, when they act at a point.

Force \( A \), 10N acting at 45° from the positive horizontal axis.

Force \( B \), 87N acting at 120° from the positive horizontal axis.

Force \( C \), 15N acting at 210° from the positive horizontal axis.

The space diagram is shown in Fig. 20.10. The forces may be written as complex numbers.

Thus force \( A \), \( f_A = 10\angle45^\circ \), force \( B \), \( f_B = 8\angle120^\circ \) and force \( C \), \( f_C = 15\angle210^\circ \).

The resultant force

\[
= f_A + f_B + f_C = 10\angle45^\circ + 8\angle120^\circ + 15\angle210^\circ
\]

\[
= 10(\cos45^\circ + j\sin45^\circ) + 8(\cos120^\circ + j\sin120^\circ)
\]

\[
= (7.071 + j7.071) + (-4.00 + j6.928)
\]

\[
= (-12.99 - j7.50)
\]

Magnitude of resultant force

\[
= \sqrt{(-9.919)^2 + (6.499)^2} = 11.86 \text{ N}
\]

Direction of resultant force

\[
= \tan^{-1} \left( \frac{6.499}{-9.919} \right) = 146.77^\circ
\]

(since \(-9.919 + j6.499 \) lies in the second quadrant).

**Exercise 89.** Further problems on applications of complex numbers

1. Determine the resistance \( R \) and series inductance \( L \) (or capacitance \( C \)) for each of the following impedances assuming the frequency to be 50Hz.

(a) \( 3 + j8 \) \( \Omega \)  
(b) \( 2 - j3 \) \( \Omega \)  
(c) \( j\mu \)  
(d) \( 8\angle-60^\circ \) \( \Omega \)

\[
\begin{bmatrix}
(a) R = 3 \Omega, L = 25.5 \text{ mH} \\
(b) R = 2 \Omega, C = 1061 \mu \text{F} \\
(c) R = 0, L = 44.56 \text{ mH} \\
(d) R = 4 \Omega, C = 459.4 \mu \text{F}
\end{bmatrix}
\]
2. Two impedances, \( Z_1 = (3 + j6) \Omega \) and \( Z_2 = (4 - j3) \Omega \) are connected in series to a supply voltage of 120 V. Determine the magnitude of the current and its phase angle relative to the voltage. 

\[ 15.76 \text{ A}, \ 23.20^\circ \text{ lagging} \]

3. If the two impedances in Problem 2 are connected in parallel determine the current flowing and its phase relative to the 120 V supply voltage. 

\[ 27.25 \text{ A}, \ 3.37^\circ \text{ lagging} \]

4. A series circuit consists of a 12 \( \Omega \) resistor, a coil of inductance 0.10 H and a capacitance of 160 \( \mu \text{F} \). Calculate the current flowing and its phase relative to the supply voltage of 240 V, 50 Hz. Determine also the power factor of the circuit. 

\[ 14.42 \text{ A}, \ 43.85^\circ \text{ lagging}, \ 0.721 \]

5. For the circuit shown in Fig. 20.11, determine the current \( I \) flowing and its phase relative to the applied voltage. 

\[ 14.6 \text{ A}, \ 2.51^\circ \text{ leading} \]

6. Determine, using complex numbers, the magnitude and direction of the resultant of the coplanar forces given below, which are acting at a point. Force \( A \), 5 N acting horizontally, Force \( B \), 9 N acting at an angle of 135° to force \( A \), Force \( C \), 12 N acting at an angle of 240° to force \( A \). 

\[ 8.394 \text{ N}, \ 208.68^\circ \text{ from force } A \]

7. A delta-connected impedance \( Z_A \) is given by:

\[
Z_A = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}
\]

Determine \( Z_A \) in both Cartesian and polar form given \( Z_1 = (10 + j0) \Omega \), \( Z_2 = (0 - j10) \Omega \) and \( Z_3 = (10 + j10) \Omega \).

\[ [(10 + j20) \Omega, \ 22.36 \pm 63.43^\circ \Omega] \]

8. In the hydrogen atom, the angular momentum, \( p \), of the de Broglie wave is given by: 

\[
p\psi = -\left(\frac{j\hbar}{2\pi}\right)(\pm jm\psi).
\]

Determine an expression for \( p \).

\[
\pm \frac{m\hbar}{2\pi}
\]

9. An aircraft \( P \) flying at a constant height has a velocity of \((400 + j300)\text{ km/h}\). Another aircraft \( Q \) at the same height has a velocity of \((200 - j600)\text{ km/h}\). Determine (a) the velocity of \( P \) relative to \( Q \), and (b) the velocity of \( Q \) relative to \( P \). Express the answers in polar form, correct to the nearest km/h.

\[
\begin{align*}
\text{(a) } & 922 \text{ km/h at } 77.47^\circ \\
\text{(b) } & 922 \text{ km/h at } -102.53^\circ
\end{align*}
\]

10. Three vectors are represented by \( P, \ 2 \angle 30^\circ, \ Q, \ 3 \angle 90^\circ \) and \( R, \ 4 \angle -60^\circ \). Determine in polar form the vectors represented by (a) \( P + Q + R \), (b) \( P - Q - R \).

\[
\begin{align*}
\text{(a) } & 3.770 \angle 8.17^\circ \\
\text{(b) } & 1.488 \angle 100.37^\circ
\end{align*}
\]

11. In a Schering bridge circuit,

\[
Z_X = (R_X - jX_C), \ Z_2 = -jX_C, \ Z_3 = \frac{(R_3)(-jX_C)}{(R_3 - jX_C)} \text{ and } Z_4 = R_4
\]

where \( X_C = \frac{1}{2\pi fC} \)

At balance: \( (Z_X)(Z_3) = (Z_2)(Z_4) \).

Show that at balance \( R_X = \frac{C_3 R_4}{R_4} \) and \( C_X = \frac{C_2 R_3}{R_4} \).